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Gauge transformation approach to the exact solution of a generalized harmonic oscillator

Fu-li Li†‡, S J Wang†§, A Weiguny† and D L Lin‖

† Institut für Theoretische Physik I, Universität Münster, 48149 Münster, Federal Republic of Germany

‡ Department of Physics, Xian Jiaotong University, Xian 710049, People's Republic of China

§ Department of Modern Physics, Lanzhou University, Lanzhou 730000, People's Republic of China

‖ Department of Physics, Hong Kong University of Science and Technology, Hong Kong, and Department of Physics and Astronomy, State University of New York at Buffalo, Buffalo, NY 14260, USA

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Abstract. The exact solution and the invariant operator of a generalized harmonic oscillator are obtained by performing three consecutive gauge transformations on the time-dependent Schrödinger equation. In contrast to previous results, they depend only on solutions of a linear differential equation for the associated classical harmonic oscillator. Non-adiabatic and adiabatic Berry phases for the system are calculated on the basis of the exact solution.

As is well known, small vibrations of dynamic systems can be described in terms of harmonic oscillators in both quantum and classical mechanics. To include surrounding influences on the vibration, or to simulate the coupling of the vibration with other degrees of freedom, one can consider time-dependent parameters specifying the Hamiltonian of a harmonic oscillator, such as the mass and the frequency. Although in general the time dependent harmonic oscillator is only an approximate description of the vibration, it is the first step towards the study of more complicated motion of dynamic systems. In some particular cases, however, this description may become exact. For example, the motion of an ion in a Paul trap is precisely described by a harmonic oscillator with periodically time-dependent frequency [1]. Recent studies [2–7] have shown that a time-dependent harmonic oscillator can acquire Berry phase when its parameters undergo a cyclic change. It is therefore of great interest to study the 'generalized harmonic oscillator' or a harmonic oscillator with time-dependent mass and frequency.

There have been two approaches to this problem. One is the evolution operator method developed by Wei and Norman [8]. The other is the invariant operator method proposed by Lewis and Riesenfeld [9]. The former has the advantage that it provides a systematic way to construct an evolution operator of the time-dependent Schrödinger equation with the Hamiltonian being a linear function of operators of an algebra. It has been shown that the Hamiltonian of a generalized harmonic oscillator can be written as a linear function of operators of an su(1, 1) algebra [10–12].

Therefore, in principle, the evolution operator of the time-dependent Schrödinger equation for a generalized harmonic oscillator can be constructed in terms of solutions of a second-order differential equation [10–12]. Because the evolution operator has to be unitary and has to satisfy a special initial condition, the solution which is used to construct the evolution operator must be real and must obey corresponding initial conditions [12]. As a consequence, the evolution operator becomes a complicated functional of the solution [10, 11]. Thus the evolution operator method provides only a formal solution of the problem in general. Besides, it does not give information about invariant quantities of the time-dependent system. In contrast, the invariant operator method emphasizes that a generalized harmonic oscillator has a time-dependent Hermitian invariant operator. It has been shown that the general solution of the time-dependent Schrödinger equation for the oscillator can be expressed as a linear superposition of eigenstates of the invariant operator, and that the coefficients in the superposition are independent of time [4, 9].

Because of the existence of an invariant operator, a generalized harmonic oscillator must be integrable. For an integrable system, the Hamiltonian can be transformed into a sum of time-independent commuting operators through a series of time-dependent gauge transformations [13]. As a consequence, the general solution of the time-dependent Schrödinger equation for an integrable system can be written as a linear superposition of common eigenstates of these commuting operators. In this paper, we show that the exact solution of a generalized harmonic oscillator can be found by introducing three consecutive gauge transformations. A time-dependent invariant operator for the system appears automatically in the process. We also show that the auxiliary functions for constructing the invariant operator are just solutions of the classical velocity-dependent harmonic oscillator with time-dependent mass and frequency. On the basis of the exact solution, non-adiabatic and adiabatic Berry phases are calculated.

Consider a generalized harmonic oscillator with the Hamiltonian given by

$$\hat{H}(t)=\tfrac{1}{2}[Z(t)\hat{p}^2+Y(t)(\hat{q}\hat{p}+\hat{p}\hat{q})+X(t)\hat{q}^2] \tag{1}$$

where $X(t)$, $Y(t)$ and $Z(t)$ are non-singular and real functions of time. The system evolves in time according to the Schrödinger equation (assume $\hbar=1$)

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi(t)\rangle=\hat{H}(t)|\Psi(t)\rangle. \tag{2}$$

Suppose that $U(t)$ is a time-dependent transformation such that

$$|\Psi'(t)\rangle=U(t)^{-1}|\Psi(t)\rangle. \tag{3}$$

Substituting (3) into (2), we find the equation of motion for $|\Psi'(t)\rangle$,

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi'(t)\rangle=\left(U^{-1}\hat{H}U-\mathrm{i}U^{-1}\frac{\partial U}{\partial t}\right)|\Psi'(t)\rangle. \tag{4}$$

In the above equation, the new Hamiltonian operator

$$\hat{H}'=U^{-1}\hat{H}U-\mathrm{i}U^{-1}\frac{\partial U}{\partial t} \tag{5}$$

should be Hermitian. This requires that $U(t)$ must be a unitary operator. Since the Hamiltonian (1) of the system is dependent on time, $U(t)$ results in a gauge transformation (3) for the wavefunction and a gauge transformation (5) for the Hamiltonian so that the form of the Schrödinger equation remains gauge invariant. It should be emphasized that the fundamental operators $\hat{q}$ and $\hat{p}$ do not depend on time in the Schrödinger picture [14].

In order to solve (2) with the Hamiltonian specified by (1), we first try to remove the cross term in the Hamiltonian (1). This can be achieved by the transformation

$$U_1(t)=\exp\left[-\frac{\mathrm{i}}{2}\int_0^t Y(t')\,\mathrm{d}t'(\hat{q}\hat{p}+\hat{p}\hat{q})\right]. \tag{6}$$

It can easily be shown that under this transformation the coordinate and momentum operators change according to

$$U_1^{-1}(t)\hat{p}U_1(t)=\hat{p}\exp\left[-\int_0^t Y(t')\,\mathrm{d}t'\right] \tag{7}$$

$$U_1^{-1}(t)\hat{q}U_1(t)=\hat{q}\exp\left[\int_0^t Y(t')\,\mathrm{d}t'\right]. \tag{8}$$

Inserting (6)–(8) into (4), we have

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi'(t)\rangle=\left[\frac{1}{2m(t)}\hat{p}^2+\frac{1}{2}k(t)\hat{q}^2\right]|\Psi'(t)\rangle \tag{9}$$

where

$$m(t)=\exp\left(2\int_0^t Y(t')\,\mathrm{d}t'\right)(Z(t))^{-1} \tag{10}$$

$$k(t)=\exp\left(2\int_0^t Y(t')\,\mathrm{d}t'\right)X(t). \tag{11}$$

Next we consider the following transformations:

$$|\Psi'(t)\rangle=U_2(t)|\Psi''(t)\rangle=\exp[\mathrm{i}C_1(t)\hat{q}^2]|\Psi''(t)\rangle \tag{12}$$

$$|\Psi''(t)\rangle=U_3(t)|\Psi'''(t)\rangle=\exp[\mathrm{i}C_2(t)(\hat{p}\hat{q}+\hat{q}\hat{p})]|\Psi'''(t)\rangle \tag{13}$$

where $C_1(t)$ and $C_2(t)$ are real functions of time. These coefficients are chosen in such a way that the Hamiltonian in (9), after these transformations, becomes a product of two factors, namely a simple time-independent harmonic oscillator Hamiltonian multiplied by an overall time-dependent factor.

The equation of motion for the coordinate $q(t)$ with the classic counterpart of the Hamiltonian in (9) is

$$\ddot{q}+\frac{\dot{m}}{m}\dot{q}+\frac{k}{m}q=0 \tag{14}$$

where the dot denotes differentiation with respect to time, and m and k are defined by (10) and (11). Assume that $f(t)$ and $f^*(t)$ are two linearly independent complex conjugate solutions of (14). Substituting $f(t)$ and $f^*(t)$ into (14), it can easily be shown that they satisfy the condition

$$\frac{\mathrm{d}}{\mathrm{d}t}[m(f\dot{f}^*-\dot{f}f^*)]=0. \tag{15}$$

As a consequence, one has

$$m(f\dot{f}^*-\dot{f}f^*)=2\mathrm{i}W \tag{16}$$

where W is a real constant.

Now let us perform the transformation $U_2(t)$, which transforms the coordinate and momentum operators as

$$U_2^{-1}(t)\hat{p}U_2(t)=\hat{p}+2C_1\hat{q} \tag{17}$$

$$U_2^{-1}(t)\hat{q}U_2(t)=\hat{q}. \tag{18}$$

Substituting (12) into (9) and using (17) and (18), we find the equation of motion for $|\Psi''(t)\rangle$,

$$i\frac{\partial}{\partial t}|\Psi''(t)\rangle=\left[\frac{1}{2m}\hat{p}^2+\frac{C_1}{m}(\hat{p}\hat{q}+\hat{q}\hat{p})+\left(\dot{C}_1+\frac{2C_1^2}{m}+\frac{k}{2}\right)\hat{q}^2\right]|\Psi''(t)\rangle. \tag{19}$$

Choosing

$$C_1(t)=\frac{m(t)}{4}\left(\frac{\dot{f}}{f}+\frac{\dot{f}^*}{f^*}\right) \tag{20}$$

and utilizing (14) and (16), one can show that (19) becomes

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi''(t)\rangle=\left[\frac{1}{2m}\hat{p}^2+\frac{1}{4}\left(\frac{\dot{f}}{f}+\frac{\dot{f}^*}{f^*}\right)(\hat{p}\hat{q}+\hat{q}\hat{p})+\frac{W^2}{2m|f|^4}\hat{q}^2\right]|\Psi''(t)\rangle. \tag{21}$$

We now consider the transformation $U_3(t)$. Since this transformation is formally the same as $U_1(t)$, it is seen from (7) and (8) that $\hat{p}$ and $\hat{q}$ undergo only a scaling change under $U_3(t)$. This suggests that one can remove the cross term in (21) by taking

$$C_2(t)=-\tfrac{1}{4}\ln|f(t)|^2. \tag{22}$$

With this choice, (21) is changed under the transformation into

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi'''(t)\rangle=\frac{1}{2m|f|^2}(\hat{p}^2+W^2\hat{q}^2)|\Psi'''(t)\rangle \tag{23}$$

which is just what we are looking for. Obviously, one can write the solution of (23) as

$$|\Psi'''(t)\rangle=\exp\left[-\mathrm{i}\int_0^t\frac{\mathrm{d}t'}{2m(t')|f(t')|^2}(\hat{p}^2+W^2\hat{q}^2)\right]|\Psi'''(0)\rangle. \tag{24}$$

To get an explicit form of the solution, let us assume the initial state

$$|\Psi'''(0)\rangle=|n\rangle \tag{25}$$

which satisfies the eigenequation

$$\tfrac{1}{2}(\hat{p}^2+W^2\hat{q}^2)|n\rangle=(n+\tfrac{1}{2})W|n\rangle \qquad n=0,1,2,\ldots. \tag{26}$$

Inserting (25) into (24), we have

$$|\Psi'''(t)\rangle=\exp\left[-\mathrm{i}\left(n+\frac{1}{2}\right)W\int_0^t\frac{\mathrm{d}t'}{m(t')|f(t')|^2}\right]|n\rangle. \tag{27}$$

The exact solution of the original equation (2) can now be found by combining the above results. We finally obtain

$$
\begin{aligned}
|\Psi_n(t)\rangle &= U_1(t)U_2(t)U_3(t)|\Psi_n'''(t)\rangle \\
&= \exp\left[-\mathrm{i}\left(n+\frac{1}{2}\right)\int_0^t \frac{\beta(t')}{|f|^2}\,\mathrm{d}t'\right]\exp\left[\mathrm{i}\frac{\mathrm{Re}(\dot f f^*)}{2Z(t)|f|^2}\hat q^2\right] \\
&\quad\times\exp\left[-\frac{\mathrm{i}}{2}\left(\frac{1}{2}\ln|f|^2+\int_0^t Y(t')\,\mathrm{d}t'\right)(\hat p\hat q+\hat q\hat p)\right]|n\rangle
\end{aligned}
\tag{28}
$$

where

$$
\beta(t)=\mathrm{Im}(f(t)\dot f^*(t))=W/m(t). \tag{29}
$$

In q-representation, the state vector (28) takes the form

$$
\begin{aligned}
\Psi_n(q,t) &= \left[\frac{\sqrt{\beta(t)}}{2^n n!\,|f|\sqrt{\pi Z(t)}}\right]^{1/2}\exp\left[-\mathrm{i}\left(n+\frac{1}{2}\right)\int_0^t \frac{\beta(t')}{|f|^2}\,\mathrm{d}t'\right] \\
&\quad\times\exp\left[(\mathrm{i}\,\mathrm{Re}(\dot f f^*)-\beta(t))\frac{q^2}{2Z(t)|f|^2}\right]H_n\left(\sqrt{\frac{\beta(t)}{Z(t)}}\frac{q}{|f|}\right)
\end{aligned}
\tag{30}
$$

where H_n are Hermite polynomials. Now let us consider the operator

$$
\begin{aligned}
\hat I(t) &= U_1(t)U_2(t)U_3(t)\tfrac{1}{2}(\hat p^2+W^2\hat q^2)U_3^{-1}(t)U_2^{-1}(t)U_1^{-1}(t) \\
&= \frac{1}{2}\left[|f|^2\hat p^2-\frac{\mathrm{Re}(\dot f f^*)}{Z(t)}(\hat p\hat q+\hat q\hat p)+\frac{|\dot f|^2}{Z^2(t)}\hat q^2\right]\exp\left(2\int_0^t Y(t')\,\mathrm{d}t'\right).
\end{aligned}
\tag{31}
$$

It can be shown by direct substitution that $\hat I(t)$ satisfies the equation

$$
\frac{\mathrm{d}\hat I}{\mathrm{d}t}=\frac{\partial \hat I}{\partial t}+\mathrm{i}[\hat H,\hat I]=0. \tag{32}
$$

Thus $\hat I(t)$ is a time-dependent Hermitian invariant operator. Applying $\hat I(t)$ to (28) or (30), we can show that

$$
\chi_n(t)=\left[\frac{\sqrt{\beta(t)}}{2^n n!\,|f|\sqrt{\pi Z(t)}}\right]^{1/2}\exp\left[\mathrm{i}\frac{\mathrm{Re}(\dot f f^*)}{2Z(t)|f|^2}q^2\right]\exp\left[-\frac{\beta(t)}{2Z(t)|f|^2}q^2\right]H_n\left(\sqrt{\frac{\beta(t)}{Z(t)}}\frac{q}{|f|}\right) \tag{33}
$$

is an eigenstate of $\hat I(t)$ with the eigenvalue $(n+\frac{1}{2})W$. Since (30) is the solution of (2) with initial condition (25) and differs from (33) only by the time-dependent phase

$$
\alpha_n(t)=-\left(n+\frac{1}{2}\right)\int_0^t \frac{\beta(t')}{|f(t')|^2}\,\mathrm{d}t' \tag{34}
$$

the general solution of (2) can be written as

$$
\Psi(q,t)=\sum_n a_n\exp\left[-\mathrm{i}\left(n+\frac{1}{2}\right)\int_0^t \frac{\beta(t')}{|f(t')|^2}\,\mathrm{d}t'\right]\chi_n(t) \tag{35}
$$

where a_n are time-independent expansion coefficients to be determined by the initial condition $\Psi(q,0)$. Expression (35) is just what is anticipated by the Lewis–Riesenfeld theory [9]. It is clear from (31) and (33) that when a scaling transformation is performed on $f(t)$ and $f(t)^*$, the invariant operator $\hat{I}(t)$ undergoes a corresponding scaling transformation, while the eigenstates do not change. Hence the solution (35) does not depend on the scaling of the solution of (14). It has been claimed in the literature that the exact solution of (2) and the invariant operator which satisfies (32) have been found [2, 4, 5, 9]. These results, however, depend on a function of time which has to be determined by solving a nonlinear second-order differential equation. In the present case, the function $f(t)$ is only a complex solution of the linear equation (14). Furthermore, since (14) results from the classical canonical equation for the coordinate $q(t)$ and the momentum $p(t)$ with the classical counterpart of the Hamiltonian in (9), the relation between the classical and quantum motion now becomes more transparent [13].

Now let us calculate the Berry phase for the generalized harmonic oscillator from the state vector in q-representation, (30). Suppose that at $t=0$ the oscillator is in the nth eigenstate of the invariant operator $\hat{I}(t)$. It will evolve into the state specified by (30) at a later time t. If the parameters $X(t)$, $Y(t)$ and $Z(t)$ are periodic functions of time with the same period T, i.e. $X(t+T)=X(t)$, $Y(t+T)=Y(t)$ and $Z(t+T)=Z(t)$, (14) may have periodic solutions. When the auxiliary function $f(t)$ in (30) is a periodic solution of (14), i.e. $f(t+T)=f(t)$, then after one period of evolution the system returns to the intial state except for acquiring the total phase

$$\varphi_t=-\left(n+\frac{1}{2}\right)\int_0^T\frac{\beta(t')}{|f(t')|^2}\,\mathrm{d}t'. \tag{36}$$

The conventional dynamic phase obtained over one period is

$$\begin{aligned}\varphi_d&=\int_0^T\langle\Psi_n(t')|\hat{H}|\Psi_n(t')\rangle\mathrm{d}t'\\&=\frac{1}{2}\left(n+\frac{1}{2}\right)\int_0^T\frac{|\dot{f}|^2+2Y\,\mathrm{Re}(\dot{f}f^*)+ZX|f|^2}{\beta(t')}\,\mathrm{d}t'.\end{aligned} \tag{37}$$

Therefore, the non-adiabatic Berry phase [15] is given by

$$\begin{aligned}\varphi_B&=\varphi_t+\varphi_d\\&=\left(n+\frac{1}{2}\right)\int_0^T\left[-\frac{\beta(t')}{|f|^2}+\frac{|\dot{f}|^2+2Y\,\mathrm{Re}(\dot{f}f^*)+ZX|f|^2}{z\beta(t')}\right]\mathrm{d}t'.\end{aligned} \tag{38}$$

Since $\beta(t)$ is quadratic in $f(t)$ and $f^*(t)$, it is clear from (38) that the non-adiabatic Berry phase does not change under a scaling transformation of $f(t)$.

We now show that in the adiabatic limit the phase (38) recovers the expression for the Berry phase as existing in the literature [2–7]. Substituting (10) and (11) into (14), we have

$$\ddot{q}(t)+\left(2Y-\frac{\dot{Z}}{Z}\right)\dot{q}(t)+XZq(t)=0. \tag{39}$$

Setting

$$q(t)=q_0\sqrt{Z(t)}Q(t)\exp\left(-\int_0^t Y(t')\,\mathrm{d}t'\right) \tag{40}$$

and substituting it into (39), we find the equation of motion for $Q(t)$,

$$\ddot{Q}(t)+\left[XZ-Y^2-Z\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Y}{Z}\right)+\frac{\ddot{Z}}{2Z}-\frac{3}{4}\left(\frac{\dot{Z}}{Z}\right)^2\right]Q(t)=0. \tag{41}$$

If the parameters $X(t)$, $Y(t)$ and $Z(t)$ vary so slowly in time that terms involving $\ddot{Z}$ and $\dot{Z}^2$ in (41) can be neglected, we expect that $Q(t)$ takes the form

$$Q(t)=\exp\left[-\mathrm{i}\int_0^t S(t')\,\mathrm{d}t'\right] \tag{42}$$

where $S(t)$ is a slowly varying function of time. Substituting (42) into (41) and neglecting the term involving $\dot{S}$, we find

$$\begin{aligned}S(t)&=\sqrt{\omega_a^2(t)-Z\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Y}{Z}\right)}\\&=\omega_a(t)-\frac{1}{2\omega_a}Z\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Y}{Z}\right)+\dots\end{aligned} \tag{43}$$

where

$$\omega_a(t)=\sqrt{XZ-Y^2}. \tag{44}$$

In the adiabatic limit, it is sufficient to go in the expansion (43) up to the second term. Therefore, the adiabatic approximate solution of (39) is given by

$$q(t)=q_0\sqrt{Z(t)}\exp\left[-\int_0^t\left(Y+\mathrm{i}\omega_a(t')-\mathrm{i}\frac{Z}{2\omega_a(t')}\frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{Y}{Z}\right)\right)\mathrm{d}t'\right]. \tag{45}$$

Replacing $f(t)$ and $f(t)^*$ in (36) and (37) by $q(t)$ and $q(t)^*$, we obtain in the adiabatic limit the total and dynamic phases

$$\varphi_t=-\left(n+\frac{1}{2}\right)\int_0^T\left[\omega_a(t')-\frac{Z}{2\omega_a(t')}\frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{Y}{Z}\right)\right]\mathrm{d}t' \tag{46}$$

$$\varphi_d=(n+\tfrac{1}{2})\int_0^T\omega_a(t')\,\mathrm{d}t'. \tag{47}$$

Consequently, the adiabatic Berry phase is given by

$$\varphi_B=\varphi_t+\varphi_d=\left(n+\frac{1}{2}\right)\int_0^T\frac{Z}{2\omega_a(t')}\frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{Y}{Z}\right)\mathrm{d}t' \tag{48}$$

which recovers the result of [2–7]. It is noted here that the adiabatic Berry phase (48) becomes zero but the non-adiabatic one (38) still exists when the crossing term in (1) vanishes [4].

In conclusion, we have found the exact solution of the time-dependent Schrödinger equation for a generalized harmonic oscillator and the explicit expression of the time-dependent invariant operator by performing gauge transformations on the time-dependent Schrödinger equation. In contrast to existing results, the present solution and the invariant operator depend only on solutions of a linear differential equation which describes the classical harmonic oscillator with time-dependent mass and frequency. We have also found the exact expression of the non-adiabatic Berry phase for the oscillator and shown that it reduces to the usual adiabatic Berry phase when the parameters of the Hamiltonian vary slowly in time.

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